Kortfattade lösningsförslag till tentamen 764G07 del 1, 2019-08-19.

1 $f(x) = \dfrac{x^2}{x^2-4}$ $\quad \cdot\ D_f = \{x \in \mathbb{R} : x \neq \pm 2\}$

$$\lim_{x \to \mp\infty} f(x) = \lim_{x \to \mp\infty} \frac{x^2}{x^2\left(1-\frac{4}{x^2}\right)} = \lim_{x \to \mp\infty} \frac{1}{1-\frac{4}{x^2}} = 1 \Rightarrow$$

$y = 1$ är vågrät asymptot

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{x^2}{x^2-4} = \infty;\ \lim_{x \to -2^+} \frac{x^2}{x^2-4} = -\infty \Rightarrow$$

$x = -2$ är lodrät asymptot

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2}{x^2-4} = -\infty;\ \lim_{x \to 2^+} \frac{x^2}{x^2-4} = \infty \Rightarrow$$

$x = 2$ är lodrät asymptot.

$\cdot\ f'(x) = \dfrac{2x(x^2-4) - 2x \cdot x^2}{(x^2-4)^2} = -\dfrac{8x}{(x^2-4)^2} = 0 \Leftrightarrow x = 0$

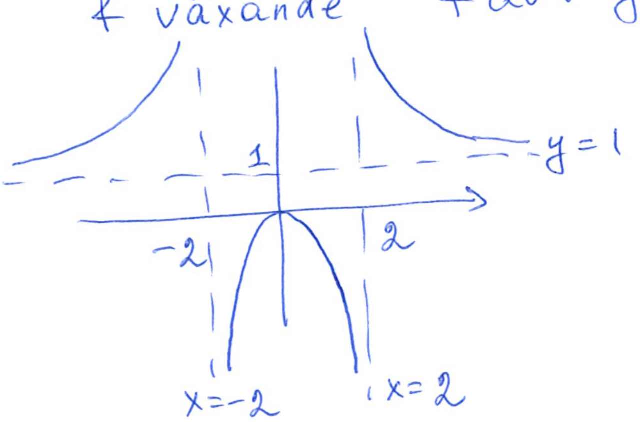
x		-2		0		2	
f'	$+$		$+$		$-$		$-$
f	$\nearrow$		$\nearrow$	lok. max	$\searrow$		$\searrow$

f växande $\quad$ f avtagande

$x = 0$ - lok. max.

$f(0) = 0$

$f(-x) = f(x) \Rightarrow$
f är jämn funktion



2 $z^5 + z^4 + 16z + 16 = 0 \Leftrightarrow z^4(z+1) + 16(z+1) = 0 \Leftrightarrow$
$(z^4+16)(z+1) = 0 \Leftrightarrow z = -1$ eller $z^4 = -16 \Leftrightarrow$
$r^4 e^{i4\varphi} = 16 e^{i\pi} \Leftrightarrow r^4 = 16$ och $4\varphi = \pi + 2\pi k \Leftrightarrow$
$r = 2$ och $\varphi = \dfrac{\pi + 2\pi k}{4}$, $k = 0, 1, 2, 3$. Alltså

$$z_1 = 2e^{i\frac{\pi}{4}} = 2\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right) = 2\left(\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$
$$z_2 = 2e^{i\frac{3}{4}\pi} = 2\left(\cos\tfrac{3}{4}\pi + i\sin\tfrac{3}{4}\pi\right) = 2\left(-\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2}$$
$$z_3 = 2e^{i\frac{5}{4}\pi} = 2\left(\cos\tfrac{5}{4}\pi + i\sin\tfrac{5}{4}\pi\right) = 2\left(-\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$$
$$z_4 = 2e^{i\frac{7}{4}\pi} = 2\left(\cos\tfrac{7}{4}\pi + i\sin\tfrac{7}{4}\pi\right) = 2\left(\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2}\right) = \sqrt{2} - i\sqrt{2}$$

Svar: $z_1 = \sqrt{2} + i\sqrt{2}$, $z_2 = -\sqrt{2} + i\sqrt{2}$, $z_3 = -\sqrt{2} - i\sqrt{2}$,
$z_4 = \sqrt{2} - i\sqrt{2}$, $z_5 = -1$.

3a

$$\lim_{x\to 2} \frac{x^3 - 2x^2 + x - 2}{x^2 + x - 6} = \lim_{x \to 2} \frac{x^2(x-2) + (x-2)}{(x-2)(x+3)} =$$

$$\lim_{x\to 2} \frac{x^2+1}{x+3} = 1$$

3b

$$\lim_{x\to 0} \frac{e^{3x}-1}{\ln(1+5x)} = \lim_{x\to 0} \underbrace{\frac{e^{3x}-1}{3x}}_{\to 1} \cdot \frac{3x}{5x} \cdot \frac{1}{\underbrace{\frac{\ln(1+5x)}{5x}}_{\to 1}} = \frac{3}{5}$$

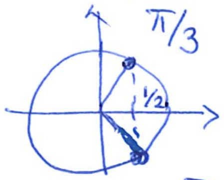
3c

$$\lim_{x\to\infty} \frac{\ln(x e^{3x})}{x + \ln x} = \lim_{x\to\infty} \frac{\ln x + 3x}{x + \ln x} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\left(\frac{\ln x}{x}^{\to 0} + 3\right)}{\cancel{x}\left(1 + \frac{\ln x}{x}\right)} = 3$$

Svar a) 1 b) $\frac{3}{5}$ c) 3.

4a $\cos\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$



$\Rightarrow 2x + \frac{\pi}{6} = \pm\frac{\pi}{3} + 2\pi k$
$k \in \mathbb{Z}$

$\Leftrightarrow 2x = \pm\frac{\pi}{3} - \frac{\pi}{6} + 2\pi k$

$x = \frac{\pi}{12} + \pi k$ eller $x = -\frac{\pi}{4} + \pi k$, $k \in \mathbb{Z}$. — svar

4b $\ln(4 - x^2) - \ln(x+2) = \ln(x^2 + x + 2)$

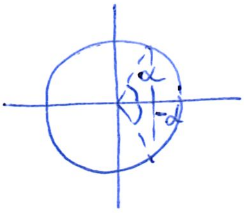
$$\begin{cases} 4 - x^2 > 0 \\ x + 2 > 0 \\ x^2 + x + 2 > 0 \end{cases} \Leftrightarrow \begin{cases} (2-x)(2+x) > 0 \\ x + 2 > 0 \\ (x+\frac{1}{2})^2 + \frac{7}{4} > 0 \end{cases} \Leftrightarrow \begin{cases} 2 - x > 0 \\ 2 + x > 0 \\ x \in \mathbb{R} \end{cases} \Leftrightarrow |x| < 2$$

$$\ln \frac{(2-x)\cancel{(2+x)}}{\cancel{x+2}} = \ln(x^2 + x + 2) \Leftrightarrow 2 - x = x^2 + x + 2$$

$\Leftrightarrow x^2 + 2x = 0 \Leftrightarrow x(x+2) = 0 \Leftrightarrow$ $x = 0$ – svar
$x = -2$ – falsk rot

4c $\quad \sin 3x = \cos 2x$

$\sin 3x = \cos\left(\frac{\pi}{2} - 3x\right) \Rightarrow \cos\left(\frac{\pi}{2} - 3x\right) = \cos 2x \Leftrightarrow$



$\frac{\pi}{2} - 3x = 2x + 2\pi n, \quad n \in \mathbb{Z}$ eller $\Leftrightarrow$

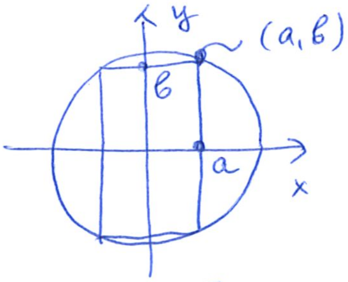
$\frac{\pi}{2} - 3x = -2x + 2\pi k, \quad k \in \mathbb{Z}$

$5x = \frac{\pi}{2} - 2\pi n$ eller $x = \frac{\pi}{2} - 2\pi k, \quad k \in \mathbb{Z}$

$\Updownarrow$

$x = \frac{\pi}{10} - \frac{2}{5}\pi n, \quad n \in \mathbb{Z}$ — Svar

5

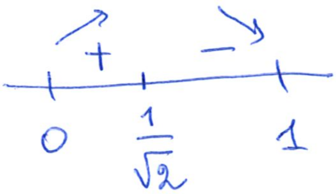


$A = 2a \cdot 2b = 4ab$ med $0 \le a, b \le 1$ samt punkten (a, b) ligger på cirkeln dvs $a^2 + b^2 = 1 \Rightarrow$

$b = \sqrt{1-a^2}$. Alltså $A = 4a\sqrt{1-a^2}, \quad 0 \le a \le 1$

$A' = 4\sqrt{1-a^2} - 4a\frac{a}{\sqrt{1-a^2}} = 4 \cdot \frac{1-a^2-a^2}{\sqrt{1-a^2}} = \frac{4(1-2a^2)}{\sqrt{1-a^2}}$

$A' = 0 \Leftrightarrow 1 - 2a^2 = 0 \Leftrightarrow a = \frac{1}{\sqrt{2}}$ ty $a > 0$.



$A\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}}\sqrt{1 - \frac{1}{2}} = \frac{4}{2} = 2$

lok. max.

Svar $\quad A_{max} = 2$ för $a = b = \frac{1}{\sqrt{2}}$.

6 $\quad f(x) = \begin{cases} x^3 + x^2, & x \le 1 \\ ax^2 + bx, & x > 1 \end{cases}$. $\quad f'(x) = \begin{cases} 3x^2 + 2x, & x < 1 \\ 2ax + b, & x > 1 \end{cases}$

Man behöver undersöka punkten $x = 1$ i övriga punkter är f deriverbar.
För att f blir deriverbar i punkten $x = 1$ bör $f(1^-) = f(1^+)$ samt $f'_-(1) = f'_+(1)$ dvs

$$\begin{cases} 2 = a + b \\ 5 = 2a + b \end{cases} \iff \begin{matrix} a = 3 \\ b = -1 \end{matrix}$$

Svar f är deriverbar om $a = 3$, $b = -1$

7 $\quad 2\ln x - \frac{1}{2}\ln(1+x^2) = 3\arctan x$

Sätt $f(x) = 2\ln x - \frac{1}{2}\ln(1+x^2) - 3\arctan x$

• D_f: $x > 0$ • $\lim\limits_{x\to 0^+} f(x) = -\infty$, $\lim\limits_{x\to\infty} f(x) =$

$= \lim\limits_{x\to\infty}\left(\ln\frac{x^2}{\sqrt{1+x^2}} - 3\arctan x\right) = \lim\limits_{x\to\infty}\left(\frac{x^2}{x\sqrt{\frac{1}{x^2}+1}} - 3\arctan x\right) = \infty$

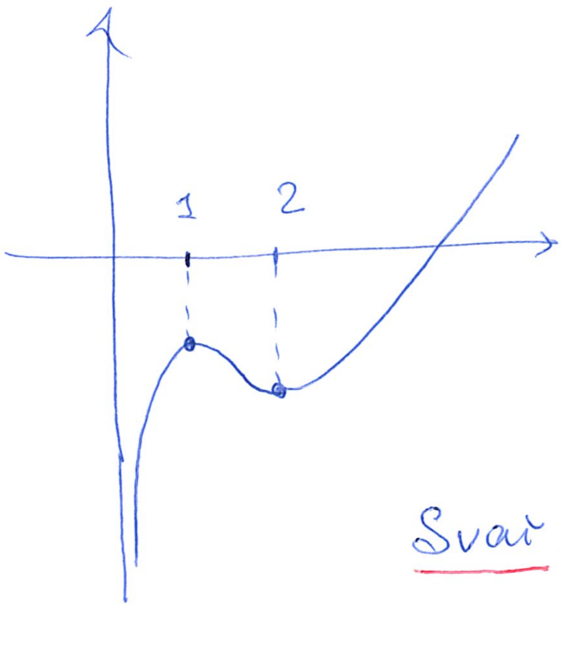
• $f'(x) = \frac{2}{x} - \frac{x}{1+x^2} - \frac{3}{1+x^2} = \frac{2(1+x^2) - x(x+3)}{x(1+x^2)} = \frac{x^2-3x+2}{x(1+x^2)}$

$= \frac{(x-1)(x-2)}{x(1+x^2)} = 0 \iff x = 1, \; x = 2$

x	0		1		2	
f'		+		−		+
f		↗	lok. max	↘	lok. min	↗

$f_{l.max} = f(1) = -\frac{1}{2}\ln 2 - \frac{3}{4}\pi$

$f_{l.min} = f(2) = \ln\frac{4}{\sqrt{5}} - 3\arctan 2$



$f(2) < f(1) < 0$

$f(x) \to \infty$ då $x \to \infty$ och f är strängt växande för $x > 2$. Alltså $f(x) = 0$ har en reell rot

Svar: ekvationen har en reell rot.